Assignment 8

Coverage: 16.2, 16.3 in Text.

Exercises: 16.2 no 10, 12, 15, 21, 22, 25, 27, 29, 30, 32, 36, 43, 46. 16.3 no 29, 31, 32.

Hand in 16.2 no 36, 43; 16.3 no 31 by March 23.

Supplementary Problems

1. A region is called star-shaped if there is a point O inside so that the line segment connecting any point in this region to O lies completely in this region. Show that the compatibility condition (3.8) is also sufficient for the existence of a potential for the vector field $\mathbf{F}$ in a star-shaped region. Hint: Modify the proof of Theorem 3.4 slightly.

Exercises 16.2

Work, Circulation, and Flux in the Plane

36. Flux across a triangle Find the flux of the field **F** in Exercise 35 outward across the triangle with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$.

Vector Fields in the Plane

43. Unit vectors pointing toward the origin Find a field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ in the xy-plane with the property that at each point $(x, y) \neq (0, 0)$, **F** is a unit vector pointing toward the origin. (The field is undefined at $(0, 0)$.)

Exercises 16.3

Applications and Examples

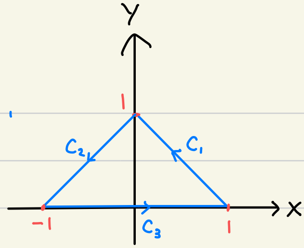
31. Evaluating a work integral two ways Let $\mathbf{F} = \nabla(x^3y^2)$ and let C be the path in the xy-plane from $(-1, 1)$ to $(1, 1)$ that consists of the line segment from $(-1, 1)$ to $(0, 0)$ followed by the line segment from $(0, 0)$ to $(1, 1)$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ in two ways.

a. Find parametrizations for the segments that make up C and evaluate the integral.

b. Use $f(x, y) = x^3y^2$ as a potential function for **F**.

§16.2

Q36 $\vec{F}(x,y) = (x+y)\vec{i} - (x^2+y^2)\vec{j} = M(x,y)\vec{i} + N(x,y)\vec{j}$.



Write $C = C_1 \cup C_2 \cup C_3$, where

· $C_1: \vec{r}(t) = (1-t)\vec{i} + t\vec{j},\ 0 \le t \le 1;\ \vec{r}\,'(t) = -\vec{i} + \vec{j}$.

$$M(\vec{r}(t))\cdot dy(t) - N(\vec{r}(t))\cdot dx(t) = [(1-t+t)\cdot 1 - (-[(1-t)^2+t^2])\cdot(-1)]dt$$

$$= [1-(1-2t+2t^2)]dt = (2t-2t^2)dt$$

· $C_2: \vec{r}(t) = -t\vec{i} + (1-t)\vec{j},\ 0 \le t \le 1;\ \vec{r}\,'(t) = -\vec{i} - \vec{j}$.

$$M(\vec{r}(t))\cdot dy(t) - N(\vec{r}(t))\cdot dx(t) = [(-t+1-t)\cdot(-1) - (-[(-t)^2+(1-t)^2])\cdot(-1)]dt$$

$$= [2t-1-(1-2t+2t^2)]dt = [-2t^2+4t-2]dt.$$

· $C_3: \vec{r}(t) = (-1+2t)\vec{i},\ 0 \le t \le 1;\ \vec{r}\,'(t) = 2\vec{i}$.

$$M(\vec{r}(t))\cdot dy(t) - N(\vec{r}(t))\cdot dx(t) = [0 - (-[(-1+2t)^2])\cdot(2)]dt = [2-8t+8t^2]dt$$

$$\therefore \text{Flux} = \int_C \vec{F}\cdot\vec{n}\,ds = \int_0^1 [(2t-2t^2)+(-2t^2+4t-2)+(2-8t+8t^2)]dt$$

$$= \int_0^1 (4t^2-2t)dt = \left[\frac{4t^3}{3} - t^2\right]_0^1 = \frac{1}{3} \ /\!/$$

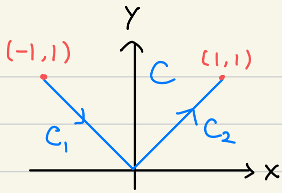
Q43 $\vec{F}(x,y) = M(x,y)\vec{i} + N(x,y)\vec{j} = \lambda(x,y)(-x\vec{i} - y\vec{j})$, where

$\lambda: \mathbb{R}^2 \setminus \{0,0\} \to \mathbb{R}$ satisfies $\overbrace{\lambda(x,y) > 0}^{(\because\ \vec{F}\text{: pointing towards the origin})}$ and $\overbrace{|\vec{F}(x,y)| = \lambda(x,y)\,|-x\vec{i} - y\vec{j}| = 1}^{(\because\ \vec{F}\text{: unit vector})}$.

$\therefore \lambda(x,y) = \dfrac{1}{\sqrt{x^2+y^2}}$. $\therefore F(x,y) = -\dfrac{x}{\sqrt{x^2+y^2}}\vec{i} - \dfrac{y}{\sqrt{x^2+y^2}}\vec{j} \ /\!/$

§16.3

Q31 (a) $\vec{F}(x,y) = \nabla(x^3y^2) = 3x^2y^2\,\vec{i} + 2x^3y\,\vec{j}$.



Write $C = C_1 \cup C_2$, where

- C_1: $\vec{r}(t) = (t-1)\vec{i} + (1-t)\vec{j}$, $0 \le t \le 1$; $\vec{r}\,'(t) = \vec{i} - \vec{j}$.

$\vec{F}(\vec{r}(t)) = 3(t-1)^2(1-t)^2\vec{i} + 2(t-1)^3(1-t)\vec{j} = 3(t-1)^4\vec{i} - 2(t-1)^4\vec{j}$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = 3(t-1)^4 \cdot 1 + (-2(t-1)^4)\cdot(-1) = 5(t-1)^4$.

- C_2: $\vec{r}(t) = t\,\vec{i} + t\,\vec{j}$, $0 \le t \le 1$; $\vec{r}\,'(t) = \vec{i} + \vec{j}$.

$\vec{F}(\vec{r}(t)) = 3t^2 \cdot t^2\,\vec{i} + 2t^3 \cdot t\,\vec{j} = 3t^4\,\vec{i} + 2t^4\,\vec{j}$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = 3t^4 \cdot 1 + 2t^4 \cdot 1 = 5t^4$.

$$\therefore \text{Work} = \int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$$

$$= \int_0^1 (5(t-1)^4 + 5t^4)\,dt$$

$$= \left[(t-1)^5 + t^5\right]_0^1 = [(0+1)-(-1)] = 2 /\!/$$

(b) $\text{Work} = \int_C (\nabla f)\cdot d\vec{r} = f(1,1) - f(-1,1) = 1 - (-1)^3 = 2 /\!/$